Stabilization for a Chain of Saturating Integrators Arising in the Visual Landing of Aircraft with Sampling

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Abstract

We provide a new output feedback control design for a chain of saturating integrators with imprecise output measurements where the outputs can also contain delays and sampling. Using a backstepping approach that leads to pointwise delays in the control and a dynamic extension, we prove input-to-state stability using a new dynamic extension approach. We utilize our main result to solve a problem in the visual landing of aircraft in the glide phase in the presence of delayed and sampled image processing.

1 Introduction

This paper continues our development of more effective output feedback stabilization methods for cases where only delayed, imprecise, or sampled output measurements are available. We began by providing a novel backstepping approach in [16] and [18], where pointwise delays were present in the feedback even if current state values are available. Our work [13, 14] then used the preceding backstepping approach to solve an output feedback control problem for a chain of saturated integrators with imprecise output measurements using an unbounded control and sampling. In the present work, we utilize our backstepping approach to solve a more challenging output stabilization problem for a chain of saturating integrators with imprecise measurements, output delays, and output sampling using a new dynamical extension; see the end of Section 2 and Section 3.1 for detailed discussions on the potential advantages of this work compared with [13, 14].

The work in this paper is a new development in a long history of research on stabilization under delays, bounded controls, and sampling. Some earlier work on bounded feedback controls includes the semi-global state and output feedback stabilization results [27], which employ linear control laws inside saturations. For some linear and nonlinear systems, crucial regional [4] stability results were presented in [25] (using LMI methods [3, 26]). Earlier bounded backstepping and forwarding methods lead to globally asymptotically stabilizing controls for some nonlinear systems; see [15] for bounded backstepping, and see [21] and [24] for forwarding methods. The delay systems literature consists largely of emulation methods (where the feedback control is designed without taking the delays in the state or output observations into account, and where one then studies the effects of state or output delays on the performance of the feedback control) and either reduction or prediction methods (which both use information about the measurement delays in the control design). However, a potential challenge in implementing standard prediction methods is that the methods usually lead to distributed terms in controls, which are terms involving an integral of past control values (but see [17] for sequential predictor or other alternative prediction methods that are free of distributed terms).

The backstepping designs from [16] and [18] circumvent the problem of determining Lie derivatives of the fictitious controls by introducing artificial delays in the control (which are called artificial because they are present even if current state values are available for measurement), and therefore are significantly different

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from prior backstepping methods. The artificial delays approach removes the smoothness requirement on the fictitious control that was present in previous backstepping approaches. The advantages of [16] are also present in the present work, which adapts the approach from [16] to solve a control problem for a chain of saturating integrators for dynamics with outputs that occurs in the vision based [8] landing of aircraft. Since only imprecise, delayed, or sampled measurements of the two first state variables are available in this application, the regional or semi-global results mentioned above do not apply, nor can we apply [16] or extensions such as [18]. This motivates our new control, which is inspired by the forwarding theory from [20]. Our controls in the present work ensure input-to-state stability with respect to additive uncertainties in the output measurements using a saturated output feedback. Also, for a positive constant $\bar{u}$, we can ensure that our control is bounded by $\bar{u}$. This contrasts with our prior work [13, 14] on chains of saturating integrators, where no amplitude constraints on the output feedback control could be satisfied and where the overshoot in the input-to-state stability estimate depends on both the additive uncertainty on the control and the maximum delay in the output measurements. See Section 3.1 for more discussion on the connections between this work and [13].

This paper improves on our preliminary conference version [1], which did not allow sampling or delays in the outputs and did not include complete proofs. Allowing delays or sampling in the outputs is motivated by the image processing in visual landing problems; see our illustration section below. However, methods such as those in [19], [22], or [23] for quantifying the effects of sampling in feedback controls would not apply here, in part because of the saturations in the dynamics and imprecise output measurements (with a multiplicative uncertainty) which place our dynamics outside the scope of existing methods for control affine systems.

The notation will be simplified whenever no confusion would arise given the context. Given any constant $T > 0$, $C_{\rm in}$ denotes the set of all continuous functions $\phi : [-T, 0] \to \mathbb{R}^n$, which we call the set of all *initial functions*. We define $\Xi_t \in C_{\rm in}$ by $\Xi_t(s) = \Xi(t+s)$ for all Ξ, $s \leq 0$, and $t \geq 0$ such that $t+s$ in the domain of Ξ. The usual Euclidean norm and the corresponding matrix norm are denoted by $|\cdot|$, and $|\cdot|_{\mathcal{S}}$ (resp., $|\cdot|_\infty$) denotes the corresponding supremum over any set $\mathcal{S}$ (resp., essential supremum). For each constant $L > 0$, we use the usual saturation function $\mathrm{sat}_L(x) = \max\{-L, \min\{L, x\}\}$. We use the standard definitions of input-to-state stability and class $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions, as defined in [9, Chapter 4], and $\mathcal{M}$ is the set of all functions of the form $\gamma + c$ where $\gamma \in \mathcal{K}_\infty$ and $c \geq 0$ is a constant. Let z_i denote the i-th entry of any vector z for each index i.

2 Problem Statement

The following system plays a valuable role in the study of the visual landing of aircraft:

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & \mathrm{sat}_{L_1}(x_2) \\ \dot{x}_2 & = & \mathrm{sat}_{L_2}(x_3) \\ \dot{x}_3 & = & \mathrm{sat}_{L_3}(u), \end{array}\right. \tag{1}$$

where $x = (x_1, x_2, x_3)$ is valued in $\mathbb{R}^3$, the input u is valued in $\mathbb{R}$, and $L_i > 0$ is a known constant for $i = 1, 2, 3$. The available output measurements are

$$\begin{array}{rcl} y_1(t) & = & \eta(t)x_1(\sigma(t)) + \delta_1(t) \\ y_2(t) & = & x_2(\sigma(t)) + \delta_2(t) \\ y_3(t) & = & x_3(t), \end{array} \tag{2}$$

where δ_1, δ_2 and η are unknown but piecewise continuous functions for which there are known constants $\overline{\eta} > 1$, $\overline{\delta}_1 \geq 0$, and $\overline{\delta}_2 \geq 0$ such that for all $t \geq 0$, we have

$$\eta(t) \in [1, \overline{\eta}] \;\text{ and }\; |\delta_i(t)| \leq \overline{\delta}_i \;\text{ for }\; i = 1, 2 \tag{3}$$

and the known piecewise continuous nondecreasing right continuous function $\sigma : \mathbb{R} \to \mathbb{R}$ admits a constant $\overline{\sigma} \geq 0$ such that $t - \overline{\sigma} \leq \sigma(t) \leq t$ for all $t \geq 0$ and so can model measurement delays and sampling, e.g., by taking σ of the form $\sigma(t) = t_i - \bar{\sigma}_a$ for all $t \in [t_i, t_{i+1})$ and $i \geq 0$, where the times t_i are such that $t_0 = 0$ and such that there is a constant $\epsilon_0 > 0$ such that $\epsilon_0 \leq t_{i+1} - t_i \leq \overline{\sigma}_b$ for all $i \geq 0$, and where the nonnegative constants $\bar{\sigma}_a$ and $\bar{\sigma}_b$ are such that $\bar{\sigma}_a + \bar{\sigma}_b \leq \bar{\sigma}$; see Section 4. In particular, the delayed state components x_1 and x_2 are not assumed to be available for measurement (because in addition to delays, our y_1 and y_2 formulas allow sampling and additive or multiplicative uncertainty), so the available measurements are not simply the delayed states. This justifies calling (2) outputs, and calling our feedbacks output feedbacks. The assumption that x_3 is available for measurement is justified because only the first two state components are subjected to the sampling and uncertainties from image processing in the aerospace application; see Section 4. The additive uncertainties δ_i can be used to model the effects of uncertainty in the sample times t_i or in the delay $\bar{\sigma}_a$.

Given an appropriate positive constant $\bar{u}$, our goal is to design a control u that is valued in $[-\bar{u}, \bar{u}]$ and that can be computed from the outputs (2) and that renders (1) input-to-state stable with respect to $\delta = (\delta_1, \delta_2)$. Choosing $\bar{u} \in (0, L_3)$ allows us to avoid the saturation in (1). Since the state space for (1) is $\mathbb{R}^3$, this implies as a special case that when $\delta = 0$, all solutions of (1) for all constant initial states $x(0) \in \mathbb{R}^3$ will converge to 0 as $t \to \infty$. Also, the input-to-state stability estimate will hold for all choices of the initial state. Our control will be a dynamic one that can be expressed the form $U(y_t, z_t)$, where the state $z(t)$ of the dynamic extension is computed using values y of the output, so we use an output feedback control. This contrasts with [14], which also studied (1) with the outputs (2), because in [14], the control was not required to be bounded, and also, [14] required the more stringent condition $\bar{\delta}_2 < L_1(1-e^{-1})^2/(40(1+2e^{-1}+e^{-2}))$ and [14] only proved the weaker conclusion that the closed loop system was input-to-state stable with respect to $(\delta_1, \delta_2, \sigma)$, which produced a positive overshoot in the stability estimate even if the δ_i's are zero. See also Section 4 for more on how this work is less restrictive than [14]. Hence, this paper provides potential advantages over [14], using a new dynamic extension which was not present in [14].

Requiring $\eta(t) \geq 1$ for all $t \geq 0$ is not restrictive because in practice, η will have known positive upper and lower bounds, and then we can divide the formula for the output component y_1 in (2) by $\inf\{\eta(\ell) : \ell \geq 0\}$ so the rescaled η and δ_1 are such that the rescaled η is bounded below by 1. Also, the outputs (2) are equivalent to assuming that the available measurements are $y_1(t) = \eta(\sigma(t))x_1(\sigma(t)) + \delta_1(t)$, $y_2(t) = x_2(\sigma(t)) + \delta_2(t)$, and $y_3(t) = x_3(t)$, because when the σ is also present in η, then we can define the new uncertainty $\eta_{\rm new}(t) = \eta(\sigma(t))$ to obtain outputs of the form (2). Throughout this work, we assume that the initial functions are constant at the initial time $t_0 = 0$, so $x_i(t) = x_i(0)$ for all $t \leq 0$ for $i = 1, 2, 3$ and similarly for the other states.

3 Stabilization of the System (1)

3.1 Statement of Main Result

This section provides formulas for our output feedback control that we described in Section 2. Our construction improves on the main result from [13], because [13] did not allow sampling or delays in the outputs, and because [13] did not provide a way to satisfy input constraints on u, and because [13] only asserted a weaker ultimate boundedness result (without proof) and in particular did not prove convergence to the equilibrium when $\delta = 0$. Given the positive constant saturation levels L_1, L_2, and L_3 from (1), and the constant $\bar{\sigma} \geq 0$ from our requirement on σ from our output measurements (2), and the given bounds $\bar{\eta}$, $\bar{\delta}_1$, and $\bar{\delta}_2$ from our conditions (3), our control design will introduce several constants. These additional constants p_1, p_2, T, k, L_4, $\overline{\alpha}_1$, and $\overline{\beta}_1$ will be required to satisfy additional conditions (7); see Section 4 for an illustration where (7) are satisfied, and Section 3.2 about the existence of solutions, and about how our conditions simplify in the significant special case where $\sigma(t) = t$ which is the case where there is no sampling or delays in the outputs.

Theorem 1. *Consider the system with outputs (1)-(2). Let the constants $\bar{\sigma} \geq 0$, $\bar{\delta}_1 \geq 0$, $\bar{\delta}_2 \geq 0$, $L_1 > 0$, $L_2 > 0$, $L_3 > 0$, and $\bar{\eta} > 0$ satisfy the requirements from Section 2. Choose the control*

$$
\begin{aligned}
u(t) &= \left\{ \begin{array}{ll} 0, & t \in [0, 2T) \\ -\mathrm{sat}_{L_4}\big(x_3(t) - v_1(z_t) - \beta(t)\big) + v_2(y_t, z_t) + v_3(y_t, z_t), & t \geq 2T \end{array} \right. , \quad \textit{where} && \text{(4a)}\\
v_1(z_t) &= \tfrac{k}{(1-e^{-kT})^2}\big(z_1(t) - z_2(t) - 2e^{-kT}z_1(t-T) + e^{-kT}z_2(t-T) + e^{-2kT}z_1(t-2T)\big), && \text{(4b)}\\
v_2(y_t, z_t) &= \tfrac{k^2}{(1-e^{-kT})^2}\big(-2z_1(t) + z_2(t) + 4e^{-kT}z_1(t-T) - e^{-kT}z_2(t-T) - 2e^{-2kT}z_1(t-2T) \\
&\quad + \phi(t) - 2e^{-kT}\phi(t-T) + e^{-2kT}\phi(t-2T)\big), \quad \textit{and} && \text{(4c)}\\
v_3(y_t, z_t) &= \tfrac{k}{1-e^{-kT}}\big(-z_3(t) + e^{-kT}z_3(t-T) + \omega(t) - e^{-kT}\omega(t-T)\big) && \text{(4d)}
\end{aligned}
$$

where the z_i's are the states of the dynamic extension

$$
\left\{ \begin{array}{lcl} \dot{z}_1(t) &=& k[-z_1(t) + \phi(t)] \\ \dot{z}_2(t) &=& k[-z_2(t) + z_1(t) - e^{-kT}z_1(t-T)] \\ \dot{z}_3(t) &=& k[-z_3(t) + \omega(t)] \end{array} \right. \tag{5}
$$

with the initial condition $z(\ell) = 0$ for all $\ell \in [-2T - \overline{\sigma}, 0]$, and where

$$\phi(t) = -\frac{\overline{\alpha}_1}{p_1}\mathrm{sat}_{p_1}\big(y_1(t)\big) \ \ \textit{and} \ \ \omega(t) = -\frac{\overline{\beta}_1}{p_2}\mathrm{sat}_{p_2}\big(y_2(t) - \alpha(\sigma(t))\big) \ \ \textit{and} \tag{6a}$$

$$\alpha(t) \ = \ \tfrac{z_2(t) - e^{-kT}z_2(t-T)}{(1-e^{-kT})^2} \quad \textit{and} \quad \beta(t) \ = \ \tfrac{z_3(t) - e^{-kT}z_3(t-T)}{1-e^{-kT}}, \tag{6b}$$

and where the positive constants p_1, p_2, T, k, L_4, $\overline{\alpha}_1$, and $\overline{\beta}_1$ and the model parameters are assumed to satisfy

$$\overline{\beta}_1 + \overline{\alpha}_2 < L_2 \tag{7a}$$

$$\overline{\sigma}\overline{\beta}_1 + 2\overline{\alpha}_1 + \bar{\delta}_2 < L_1 \tag{7b}$$

$$L_4 + \overline{\alpha}_3 + \overline{\beta}_2 < L_3 \tag{7c}$$

$$\overline{\eta}(\overline{\sigma} + T)\overline{\alpha}_1 + \bar{\delta}_1 < p_1 \tag{7d}$$

$$\overline{\alpha}_1\overline{\eta}^2(2T + \overline{\sigma}) < p_1 \tag{7e}$$

$$(\overline{\sigma} + T)\overline{\beta}_1 + \bar{\delta}_2 < p_2 \tag{7f}$$

$$2\overline{\beta}_1(T + \overline{\sigma}) < p_2 \tag{7g}$$

$$\frac{\overline{\alpha}_1\overline{\eta}}{p_1}(2T + \overline{\sigma})\left(\overline{\alpha}_1 + \xi_\star + \frac{\overline{\alpha}_1}{p_1}\bar{\delta}_1\right) + \xi_\star + \frac{\overline{\alpha}_1}{p_1}\bar{\delta}_1 < \frac{\overline{\alpha}_1}{\overline{\eta}} \tag{7h}$$

where

$$\xi_\star = \frac{\bar{\delta}_2\left(\overline{\beta}_1(T + \overline{\sigma}) + p_2\right)}{p_2\sqrt{1 - \frac{2}{p_2}\overline{\beta}_1(T + \overline{\sigma})}}, \ \ \overline{\alpha}_2 = \frac{2k\overline{\alpha}_1}{1 - e^{-kT}}, \ \ \overline{\alpha}_3 = \frac{4k^2\overline{\alpha}_1}{(1 - e^{-kT})^2}, \ \ \textit{and} \ \ \overline{\beta}_2 = \frac{2k\overline{\beta}_1}{1 - e^{-kT}}. \tag{8}$$

Then the following conclusions hold: (a) The control u in (4a) is bounded by $L_4 + \overline{\alpha}_3 + \overline{\beta}_2$ and (b) the system (1) in closed loop with the control (4a) satisfies an input-to-state stability estimate with respect to $\delta = (\delta_1, \delta_2)$ for all initial states $x(0) \in \mathbb{R}^3$.

3.2 Consistency of Conditions (7), Existence of Solutions, and Special Cases

This section explains how to satisfy (7), how (7) reduces to the type of conditions from [13] in the case $\sigma(t) = t$, and why the existence of solutions of the closed loop system is assured. For given constants $\bar{\sigma}$, L_1, L_2, L_3, $\bar{\eta}$, $\bar{\delta}_1$, and $\bar{\delta}_2$ in our theorem, we can satisfy (7) using the following three step process. First, choose any positive constants k and T. Second, choose the positive constants $\overline{\alpha}_1$, $\overline{\beta}_1$, and L_4 to be small enough so that (7a)-(7c) are satisfied, which can be done because of our assumption that $\bar{\delta}_2 < L_1$ and because of our formulas for $\bar{\alpha}_3$ and $\bar{\beta}_2$ from (8). Finally, choose the positive constants p_1 and p_2 to satisfy the remaining conditions from (7), which can be done when $\bar{\delta}_2$ is small enough, because of the way p_1 and p_2 appear in denominators in (7h) and (8); see Section 4. The existence of a unique solution for each constant initial state (such that the solution is defined for all $t \geq 0$) follows because of the linear growth of the right side of closed loop dynamics, the boundedness of the nonlinear terms, and standard existence and uniqueness results from basic theory of differential equations.

Since Theorem 1 ensures input-to-state stability, it provides robustness with respect to slight modifications of the system (such as the image processing method in the application). In the special case where there is neither sampling nor delays, we can pick $\sigma(t) = t$ for all real t and therefore $\bar{\sigma} = 0$ in (7). However, our method does not allow us to choose $T = 0$, because (4b)-(4d), (6b), and (8) would not be defined for $T = 0$, and because our choices of α and β in (6b) are essential for producing a system in new variables that lends itself to proving stability results; see (15). Hence, the only simplifications in our control when there is no sampling or delays in (2) are that (i) we replace $\sigma(t)$ by t in (2) and in our formula for the function ω in (6a) and (ii) we replace $\bar{\sigma}$ by 0 in (7)-(8). In particular, the control design does not significantly simplify in the absence of delays or sampling. Then our conditions in Theorem 1 are similar to those of [13] (and we recover conditions from [1] in the limit as $\bar{\sigma} \to 0$), but [13] only asserted a weaker ultimate boundedness result. Another important case is where the δ_i's are zero, in which case we can set $\bar{\delta}_1 = \bar{\delta}_2 = 0$ in (7) and we can conclude that the closed loop system is uniformly globally asymptotically stable on $\mathbb{R}^3$. This highlights a distinction between the delay $T > 0$ in our control design and the output delay that can be represented by our function σ in (2), which is that T is an artificial delay that is introduced in the control (and cannot be removed) and that T does not correspond to a delay in engineering system that is being modeled, while σ can model a delay in the engineering system.

3.3 Key Lemmas Needed for Proving Theorem 1

A key idea in our proof of Theorem 1 is to show that for large enough times, the saturations in our system and in our control can be removed, because their arguments will lie in the intervals in which the saturations agree with the identity function. To make this idea precise, and to help readers grasp the technical steps of our proof of Theorem 1, we first state lemmas, for which we provide proofs in the appendices below. Our first lemma is as follows, whose conclusions (9a) and (9c) can be combined to give the bound $L_4 + \overline{\alpha}_3 + \overline{\beta}_2$ on our control u:

Lemma 1. *In terms of the notation from Theorem 1, we have*

$$u(t) = -\mathrm{sat}_{L_4}\big(x_3(t) - \dot{\alpha}(t) - \beta(t)\big) + \ddot{\alpha}(t) + \dot{\beta}(t), \tag{9a}$$

$$\alpha(t) = \tfrac{k^2}{(1-e^{-kT})^2}\int_{t-T}^{t}\int_{m-T}^{m} e^{k(\ell-t)}\phi(\ell)\mathrm{d}\ell\mathrm{d}m,\ \ \beta(t) = \tfrac{k}{1-e^{-kT}}\int_{t-T}^{t} e^{k(\ell-t)}\omega(\ell)\mathrm{d}\ell, \tag{9b}$$

$$|\alpha(t)| \le \overline{\alpha}_1,\ \ |\dot{\alpha}(t)| \le \overline{\alpha}_2,\ \ |\ddot{\alpha}(t)| \le \overline{\alpha}_3,\ \ |\beta(t)| \le \overline{\beta}_1,\ \ \textit{and}\ \ |\dot{\beta}(t)| \le \overline{\beta}_2 \tag{9c}$$

for all $t \ge 2T$.

The next lemma follows from the Fundamental Theorem of Calculus, so we omit its proof:

Lemma 2. *Let k and T be any positive constants. Then the relations*

$$\int_{t-T}^{t}\int_{m-T}^{m} e^{k(\ell-t)}\mathrm{d}\ell\mathrm{d}m = \tfrac{1}{k^2}(1-e^{-kT})^2 \ \ \textit{and}\ \ \int_{t-T}^{t} e^{k(\ell-t)}\mathrm{d}\ell = \tfrac{1}{k}(1-e^{-kT}) \tag{10}$$

hold for all $t \ge 0$.

To realize our goal of eliminating the effects of the saturations, we will find class $\mathcal{M}$ functions $\mathcal{T}$ so that certain scalar variables $q(t)$ stay within suitable intervals $[-\bar{M}, \bar{M}]$ for all $t \ge \mathcal{T}(|q(T_0)|)$ and suitable times T_0, which will be useful for realizing our strategy of removing saturations in the course of our proof. The following lemma will allow us to find the required class $\mathcal{M}$ functions, and we prove this lemma in Appendix 2:

Lemma 3. *Let $\mathcal{G} : \mathbb{R}^2 \to \mathbb{R}$ be a bounded continuous function that admits positive constants g_0, T_0, and $\bar{M}$ and a function $g : \mathbb{R} \to [0, \infty)$ such that $\mathcal{G}(t, q) = g(t)q$ for all $t \ge T_0$ and all $q \in [-\bar{M}, \bar{M}]$ and such that $\mathcal{G}(t, q)\mathrm{sign}(q) \ge g_0$ holds for all $t \ge T_0$ and $q \in \mathbb{R} \setminus [-\bar{M}, \bar{M}]$. Let $\mathcal{H} : [0, \infty) \to \mathbb{R}$ be a bounded continuous function that admits constants $\bar{T} > T_0$ and $\bar{H} > 0$ such that $|\mathcal{H}(t)| \le \bar{H}$ for all $t \ge \bar{T}$ and such that $\bar{H} < g_0$. Then there exists a function $T_* \in \mathcal{M}$ such that for each C^1 solution $q : [T_0, \infty) \to \mathbb{R}$ of*

$$\dot{q}(t) = -\mathcal{G}(t, q(t)) + \mathcal{H}(t), \tag{11}$$

we have $|q(t)| \le \bar{M}$ for all $t \ge T_*(|q(T_0)|)$ and therefore also $\dot{q}(t) = -g(t)q(t) + \mathcal{H}(t)$ for all $t \ge T_*(|q(T_0)|)$.

Finally, we need this variant of Halanay's inequality, which generalizes [5, Lemma 4.2] to allow nonzero values of Δ_1, and which we prove in Appendix 3 and which is used in our Lyapunov analysis in Section 3.4.2:

Lemma 4. *Consider a continuous function $v : [-h, +\infty) \to [0, +\infty)$ with $h > 0$ being a constant. Assume that there are constants $\Delta_1 \ge 0$, $\mathfrak{c}_1$, and $\mathfrak{c}_2$ satisfying $\mathfrak{c}_1 > \mathfrak{c}_2 > 0$ such that the inequality*

$$\dot{v}(t) \le -\mathfrak{c}_1 v(t) + \mathfrak{c}_2 \sup_{m\in[t-h,t]} v(m) + \Delta_1 \tag{12}$$

is satisfied for all $t \ge 0$. Choose $c_s > 0$ to be the unique positive value such that $c_s = \mathfrak{c}_1 - \mathfrak{c}_2 e^{c_s h}$, and let $l_v > 0$ be a constant such that $l_v e^{-c_s t} > v(t) - \Delta_1/(\mathfrak{c}_1 - \mathfrak{c}_2)$ for all $t \in [-h, 0]$. Then

$$v(t) \le l_v e^{-c_s t} + \tfrac{\Delta_1}{\mathfrak{c}_1 - \mathfrak{c}_2} \tag{13}$$

holds for all $t \ge 0$.

3.4 Proof of Theorem 1

The the proof is arranged as follows. In the first step, we use a change of variables that produces a useful cascaded system with a globally asymptotically stable subsystem; see (16). In the second step, we perform a Lyapunov function analysis for this new system. In the third step, we use results from the first two steps to find useful bounds on the states of the original system. In the last step, we use linear growth properties of the closed loop system to transform the preceding estimates into the required input-to-state stability estimate.

3.4.1 First Step: System in New Variables with Asymptotically Stable Subsystem

We use the new variable $\xi = (x_1, \xi_2, \xi_3)$, where

$$\xi_2(t) = x_2(t) - \alpha(t) \text{ and } \xi_3(t) = x_3(t) - \dot{\alpha}(t) - \beta(t). \tag{14}$$

Direct calculations then transform (1) into the new system

$$\dot{x}_1(t) = \mathrm{sat}_{L_1}(\alpha(t) + \xi_2(t)), \quad \dot{\xi}_2(t) = \mathrm{sat}_{L_2}(\beta(t) + \xi_3(t) + \dot{\alpha}(t)) - \dot{\alpha}(t), \quad \dot{\xi}_3(t) = \mathrm{sat}_{L_3}(u) - \ddot{\alpha}(t) - \dot{\beta}(t). \tag{15}$$

Notice that the system (15) in closed-loop with (4a) is forward complete, by the forward completeness of our dynamic extension (5), meaning, all of its solutions are defined for all $t \geq 0$ for all constant initial functions at the initial time $t_0 = 0$ (which follows from the boundedness of the nonlinear terms in (5)). Also, (9c) and (7c) imply that $|u(t)| = |-\mathrm{sat}_{L_4}(\xi_3(t)) + \ddot{\alpha}(t) + \dot{\beta}(t)| \leq L_4 + \overline{\alpha}_3 + \overline{\beta}_2 < L_3$ holds for all $t \geq 2T$. As an immediate consequence, the system (15) in closed loop with the feedback defined in (4a) admits the representation

$$\dot{x}_1(t) = \mathrm{sat}_{L_1}(\alpha(t) + \xi_2(t)), \ \dot{\xi}_2(t) = \mathrm{sat}_{L_2}(\beta(t) + \xi_3(t) + \dot{\alpha}(t)) - \dot{\alpha}(t), \ \dot{\xi}_3(t) = -\mathrm{sat}_{L_4}(\xi_3(t)) \tag{16}$$

for all $t \geq 2T$. Since the ξ_3 subsystem of (16) is globally asymptotically stable to 0 on $\mathbb{R}$, we deduce from (7a) and the bounds (9c) that there is a class $\mathcal{M}$ function $T_b : [0, +\infty) \to [2T, +\infty)$ (depending on $\bar{\beta}_1$ and $\bar{\alpha}_2$) such that for all $t \geq T_b(|\xi(0)|)$, we have $|\beta(t) + \xi_3(t) + \dot{\alpha}(t)| \leq \overline{\beta}_1 + \overline{\alpha}_2 + |\xi_3(t)| < L_2$. Hence, we obtain the system

$$\dot{x}_1(t) = \mathrm{sat}_{L_1}(\alpha(t) + \xi_2(t)), \ \dot{\xi}_2(t) = \beta(t) + \xi_3(t) \tag{17}$$

for all $t \geq T_b(|\xi(0)|)$. Using the formulas $y_1(t) = \eta(t)x_1(\sigma(t)) + \delta_1(t)$ and $y_2(t) - \alpha(\sigma(t)) = x_2(\sigma(t)) + \delta_2(t) - \alpha(\sigma(t)) = \xi_2(\sigma(t)) + \delta_2(t)$ for our output components (which follow from (2) and (14)), and also using our formulas for ϕ and ω from (6a) and (9b), we can then rewrite (17) as

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & \mathrm{sat}_{L_1}\left(-\frac{k^2\overline{\alpha}_1}{(1-e^{-kT})^2 p_1}\int_{t-T}^{t}\int_{m-T}^{m} e^{k(\ell-t)}\mathrm{sat}_{p_1}\big(\eta(\ell)x_1(\sigma(\ell))\big)\mathrm{d}\ell\mathrm{d}m\right) + \mathcal{J}_1(t) \\ \dot{\xi}_2(t) & = & -\frac{\overline{\beta}_1 k}{p_2(1-e^{-kT})}\int_{t-T}^{t} e^{k(\ell-t)}\mathrm{sat}_{p_2}\big(\xi_2(\sigma(\ell))\big)\mathrm{d}\ell + \mathcal{J}_2(t), \end{array}\right. \tag{18a}$$

$$\begin{array}{rrcl} \text{where} & \mathcal{J}_1(t) & = & \mathrm{sat}_{L_1}\left(-\frac{k^2\overline{\alpha}_1}{(1-e^{-kT})^2 p_1}\int_{t-T}^{t}\int_{m-T}^{m} e^{k(\ell-t)}\mathrm{sat}_{p_1}\big(\eta(\ell)x_1(\sigma(\ell)) + \delta_1(\ell)\big)\mathrm{d}\ell\mathrm{d}m + \xi_2(t)\right) \\ & & & -\mathrm{sat}_{L_1}\left(-\frac{k^2\overline{\alpha}_1}{(1-e^{-kT})^2 p_1}\int_{t-T}^{t}\int_{m-T}^{m} e^{k(\ell-t)}\mathrm{sat}_{p_1}\big(\eta(\ell)x_1(\sigma(\ell))\big)\,\mathrm{d}\ell\mathrm{d}m\right) \\ \text{and} & \mathcal{J}_2(t) & = & \frac{\overline{\beta}_1 k}{p_2(1-e^{-kT})}\int_{t-T}^{t} e^{k(\ell-t)}\left[\mathrm{sat}_{p_2}\left(\xi_2(\sigma(\ell))\right) - \mathrm{sat}_{p_2}\big(\xi_2(\sigma(\ell)) + \delta_2(\ell)\big)\right]\mathrm{d}\ell + \xi_3(t) \end{array} \tag{18b}$$

for all $t \geq T_b(|\xi(0)|)$. The formulas (18a) will play an important role in what follows, because we will use the $\dot{\xi}_2(t)$ formula from (18a) to find an upper bound for $|\xi_2(t)|$ in terms of $|(\bar{\delta}_1, \bar{\delta}_2)|$ and an additional term that converges to 0, and then we will use the $\dot{x}_1(t)$ formula from (18a) to obtain an analogous bound for $|x_1(t)|$; see (31). This will lead to the desired conclusion for the system in the original variables.

By (7b), we have $\overline{\alpha}_1 < L_1$, and sat_{p_1} is bounded by p_1, so (18a) and the first equality in (10) give

$$\dot{x}_1(t) = -\frac{k^2\overline{\alpha}_1}{(1-e^{-kT})^2 p_1}\int_{t-T}^{t}\int_{m-T}^{m} e^{k(\ell-t)}\mathrm{sat}_{p_1}\big(\eta(\ell)x_1(\sigma(\ell))\big)\mathrm{d}\ell\mathrm{d}m + \mathcal{J}_1(t) \tag{19}$$

for all $t \geq T_b(|\xi(0)|)$. Finally, since saturations have the global Lipschitz constant 1, we can check that

$$|\mathcal{J}_2(t)| \leq |\xi_3(t)| + \tfrac{\overline{\beta}_1}{p_2}\overline{\delta}_2 \ \text{ and } \ |\mathcal{J}_1(t)| \leq |\xi_2(t)| + \tfrac{\overline{\alpha}_1}{p_1}\overline{\delta}_1 \ \text{ for all } \ t \geq T_b(|\xi(0)|), \tag{20}$$

by (10). We next analyze the stability properties of the system (18a).

By adding and subtracting $(\overline{\beta}_1/p_2)\mathrm{sat}_{p_2}(\xi_2(t))$ on the right side of the $\dot{\xi}_2(t)$ formula in (18a) and then using the second equality in (10), it follows that, for all $t \geq T_b(|\xi(0)|) + \overline{\sigma} + T$,

$$\dot{\xi}_2(t) = -\tfrac{\overline{\beta}_1}{p_2}\mathrm{sat}_{p_2}\left(\xi_2(t)\right) + \mathcal{R}_1(t) + \mathcal{J}_2(t), \ \text{ where} \tag{21a}$$

$$\mathcal{R}_1(t) = \frac{\overline{\beta}_1 k}{p_2(1-e^{-kT})}\int_{t-T}^{t} e^{k(\ell-t)}\left[\mathrm{sat}_{p_2}\left(\xi_2(t)\right) - \mathrm{sat}_{p_2}\left(\xi_2(t) - \int_{\sigma(\ell)}^{t}\dot{\xi}_2(s)\mathrm{d}s\right)\right]\mathrm{d}\ell. \tag{21b}$$

Moreover, by (9c) and (17), we have $|\dot{\xi}_2(t)| \leq |\xi_3(t)| + \overline{\beta}_1$. As a consequence, for all $t \geq T_b(|\xi(0)|) + T + \overline{\sigma}$, we have

$$|\mathcal{R}_1(t)| \ \leq \ \tfrac{\overline{\beta}_1 k}{p_2(1-e^{-kT})} \int_{t-T}^{t} e^{k(\ell-t)}(t-\sigma(\ell))\mathrm{d}\ell \left(|\xi_3|_{[t-T-\overline{\sigma},t]} + \overline{\beta}_1 \right) \ \leq \ \tfrac{\overline{\beta}_1}{p_2}(T+\overline{\sigma})\big(|\xi_3|_{[t-T-\overline{\sigma},t]} + \overline{\beta}_1\big) \quad (22)$$

where the last inequality in (22) used (10) and the bounds $\ell - \bar{\sigma} \leq \sigma(\ell)$ to get $t - \sigma(\ell) \leq t - \ell + \bar{\sigma} \leq T + \bar{\sigma}$ for all $\ell \in [t-T, t]$. The second inequality in (22) and the first inequality in (20) yield

$$|\mathcal{R}_1(t)| + |\mathcal{J}_2(t)| \ \leq \ \tfrac{\overline{\beta}_1}{p_2}\left[(T+\overline{\sigma})\left(|\xi_3|_{[t-T-\overline{\sigma},t]} + \overline{\beta}_1\right) + \overline{\delta}_2\right] + |\xi_3(t)| \quad (23)$$

for all $t \geq T_b(|\xi(0)|) + T + \overline{\sigma}$. Since the ξ_3 subsystem is globally asymptotically stable to 0, we deduce from (23) and (7f) that there is a class $\mathcal{M}$ function $T_c : [0, +\infty) \to [0, +\infty)$ and a constant $\delta_0 > 0$ such that

$$\sup_{\ell \geq T_c(|\xi(0)|)} \left(|\mathcal{R}_1(\ell)| + |\mathcal{J}(\ell)|\right) \leq \tfrac{\overline{\beta}_1}{p_2}\left[(T+\overline{\sigma})\overline{\beta}_1 + \overline{\delta}_2\right] + \delta_0 < \overline{\beta}_1. \quad (24)$$

Hence, by (21a), there is a class $\mathcal{M}$ function $T_d : [0, +\infty) \to [0, +\infty)$ such that $T_d(s) \geq T_b(s) + T + \bar{\sigma}$ for all $s \geq 0$ and such that $|\xi_2(t)| < p_2$ for all $t \geq T_d(|\xi(0)|)$, because $\dot{\xi}_2 > 0$ (resp., < 0) when $\xi_2(t) \leq -p_2$ (resp., $\geq p_2$). Then for all $t \geq T_d(|\xi(0)|)$, we have $\mathrm{sat}_{p_2}(\xi_2(t)) = \xi_2(t)$, so (21a) gives the reduced order system

$$\begin{cases} \dot{x}_1(t) &= -\frac{k^2\overline{\alpha}_1}{(1-e^{-kT})^2 p_1} \displaystyle\int_{t-T}^{t}\int_{m-T}^{m} e^{k(\ell-t)}\mathrm{sat}_{p_1}\big(\eta(\ell)x_1(\sigma(\ell))\big)\mathrm{d}\ell\mathrm{d}m + \mathcal{J}_1(t) \\ \dot{\xi}_2(t) &= -\frac{\overline{\beta}_1}{p_2}\xi_2(t) + \mathcal{R}_1(t) + \mathcal{J}_2(t) \end{cases} \quad (25)$$

in the new variables from (14) for all $t \geq T_d(|\xi(0)|)$. This follows by applying Lemma 3 to the system (21a), by choosing $q = \xi_2$, $\mathcal{G}(t, q) = (\overline{\beta}_1/p_2)\mathrm{sat}_{p_2}(q)$, $g(t) = \overline{\beta}_1/p_2$, $\bar{M} = p_2$, and $\mathcal{H} = \mathcal{R}_1 + \mathcal{J}_2$ in the lemma, and using the fact that $|\xi| \geq |\xi_2|$.

3.4.2 Second Step: Lyapunov Analysis for Reduced Order System (25) in New Variables

This step will use the candidate Lyapunov function $\Upsilon(\xi_2) = \frac{1}{2}\xi_2^2$. By (21b) and the second equality in (10), we get

$$|\mathcal{R}_1(t)| \ \leq \ \tfrac{\overline{\beta}_1 k}{p_2(1-e^{-kT})} \int_{t-T}^{t} e^{k(\ell-t)} \int_{\sigma(\ell)}^{t} |\dot{\xi}_2(s)|\mathrm{d}s\mathrm{d}\ell \ \leq \ \tfrac{\overline{\beta}_1}{p_2}\int_{t-T-\overline{\sigma}}^{t} |\dot{\xi}_2(s)|\mathrm{d}s \quad (26)$$

for all $t \geq T_d(|\xi(0)|)$, because sat_{p_2} has the Lipschitz constant 1. By using (18a) to upper bound $|\dot{\xi}_2(s)|$ in (26), we deduce from the second inequality in (26) that for all $t \geq T_d(|\xi(0)|)$, we have

$$\begin{aligned} |\mathcal{R}_1(t)| \ &\leq \ \tfrac{\overline{\beta}_1^2 k}{p_2^2(1-e^{-kT})} \int_{t-T-\overline{\sigma}}^{t}\int_{s-T}^{s} e^{k(\ell-s)}\mathrm{d}\ell\mathrm{d}s \sup_{m\in[t-2T-2\overline{\sigma},t]} |\xi_2(m)| + \tfrac{\overline{\beta}_1}{p_2}\int_{t-T-\overline{\sigma}}^{t} |\mathcal{J}_2(s)|\,\mathrm{d}s \\ &\leq \ \tfrac{\overline{\beta}_1^2(T+\overline{\sigma})}{p_2^2} \sup_{m\in[t-2T-2\overline{\sigma},t]} |\xi_2(m)| + \frac{\overline{\beta}_1(T+\overline{\sigma})}{p_2}\left(\frac{\overline{\beta}_1}{p_2}\overline{\delta}_2 + |\xi_3|_{[t-T-\overline{\sigma},t]}\right), \end{aligned} \quad (27)$$

where the last inequality is a consequence of the second equality in (10) and of the first inequality in (20). By using the first inequality in (20) and (27) to bound $\mathcal{R}_1$ and $\mathcal{J}_2$ from (25), we easily deduce from the asymptotic stability of the ξ_3 subsystem of (16) that for any constant $\omega_0 > 0$, we have

$$\begin{aligned} \dot{\Upsilon}(t) \ &\leq \ \tfrac{\overline{\beta}_1}{p_2}\left(-\xi_2^2(t) + \tfrac{\overline{\beta}_1(T+\overline{\sigma})}{p_2}\sup_{m\in[t-2T-2\overline{\sigma},t]}\xi_2^2(m) + |\xi_2(t)|\left[\left\{\left(\tfrac{\overline{\beta}_1}{p_2}(T+\overline{\sigma})+1\right)\overline{\delta}_2\right\} + \mu(|\xi(0)|,t)\right]\right) \\ &\leq \ \tfrac{\overline{\beta}_1}{p_2}\left(-\Upsilon(\xi_2(t)) + \tfrac{2\overline{\beta}_1(T+\overline{\sigma})}{p_2}\sup_{m\in[t-2T-2\overline{\sigma},t]}\Upsilon(\xi_2(m)) + \frac{1}{2}B_*(t)\right), \end{aligned} \quad (28)$$

along all solutions of (25) for all $t \geq T_d(|\xi(0|) + T + \overline{\sigma}$, where

$$B_*(t) = (1+\omega_0)\left(\tfrac{\overline{\beta}_1}{p_2}(T+\overline{\sigma})+1\right)^2\overline{\delta}_2^2 + \left[\left(1+\tfrac{1}{\omega_0}\right)\mu^2(|\xi(0)|,t)\right], \quad (29)$$

and where the last inequality in (28) used the triangle inequality $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ with $a = |\xi_2(t)|$ and b being the quantity in squared brackets in (28) (followed by a use of the relation $(r+s)^2 \le (1+\omega_0)r^2 + (1+(1/\omega_0))s^2$ where r is the quantity in curly braces in (28) and $s = \mu(|\xi(0)|, t)$), and where μ is a class $\mathcal{KL}$ function. Since $\mu \in \mathcal{KL}$, we can then find a class $\mathcal{M}$ function $T_d^\sharp : [0, +\infty) \to [0, +\infty)$ such that $T_d^\sharp(s) \ge T_d(s) + T$ for all $s \ge 0$ and such that $\mu(|\xi(0)|, t) \le \omega_0$ for all $t \ge T_d^\sharp(|\xi(0)|)$, and therefore such that the quantity in squared brackets in (29) is bounded above by $\omega_0 + \omega_0^2$ for all $t \ge T_d^\sharp(|\xi(0)|)$. Therefore, for any constant $\epsilon > 0$, we can choose $\omega_0 > 0$ small enough (with ω_0 depending on ϵ and the other parameters) such that

$$\sqrt{B_*(t)\overline{\beta}_1/(p_2(\mathfrak{c}_1 - \mathfrak{c}_2))} \le \left(\sqrt{1+\omega_0}\left(\frac{\bar{\beta}_1}{p_2}(T+\bar{\sigma}) + 1\right)\bar{\delta}_2 + \sqrt{\omega_0 + \omega_0^2}\right)\sqrt{\frac{\bar{\beta}_1}{p_2(\mathfrak{c}_1 - \mathfrak{c}_2)}} < \xi_\star(1+\epsilon) \tag{30}$$

for all $t \ge T_d^\sharp(|\xi(0)|)$, where $\xi_\star$ is from (8), $\mathfrak{c}_1 = \bar{\beta}_1/p_2$, $\mathfrak{c}_2 = 2\bar{\beta}_1^2(T+\bar{\sigma})/p_2^2$, and the first inequality in (30) used the subadditivity of the square root; the fact that $\mathfrak{c}_1 > \mathfrak{c}_2$ follows because (7g) implies that $1 > 2\bar{\beta}_1(T+\bar{\sigma})/p_2$. Since $1 > 2\bar{\beta}_1(T+\bar{\sigma})/p_2$, it follows from Lemma 4 (applied to $v(t) = \Upsilon(\xi_2(t + T_d^\sharp(|\xi(0)|)))$, and with the preceding choices of $\mathfrak{c}_1$ and $\mathfrak{c}_2$ and the choice $\Delta_1 = \frac{1}{2}(\bar{\beta}_1/p_2)\sup\{B_*(t) : t \ge T_d^\sharp(|\xi(0)|\})$ that

$$|\xi_2(t)| \quad \le \quad \xi_\star(1+\epsilon) + \mu_0(|x(0)| + |\delta|_\infty, t) \text{ for all } t \ge T_d^\sharp(|\xi(0)|); \tag{31}$$

this is done by using (9b) and (14) to find a $\gamma \in \mathcal{K}_\infty$ such that $|\xi(0)| \le \gamma(|x(0)| + |\delta|_\infty)$ for all initial states and also using the second equality in (17) and the linear growth of the dynamics to bound $|\xi_2|$ on $[0, T_d^\sharp(|\xi(0)|)$, in order to find the required function $\mu_0 \in \mathcal{KL}$. We also used the subadditivity of the square root to transform the upper bound on $\Upsilon(\xi_2(t))$ that is obtained from Lemma 4 into the upper bound (31) on $|\xi_2(t)|$.

The next part is devoted to the x_1-subsystem (25). First, we deduce from (25) that $|\dot{x}_1(t)| \le \overline{\alpha}_1 + |\mathcal{J}_1(t)| \le \overline{\alpha}_1 + |\xi_2(t)| + \frac{\overline{\alpha}_1}{p_1}\bar{\delta}_1$, using the first equality in (10) and the second inequality in (20), so (31) gives

$$|\dot{x}_1(t)| \quad \le \quad \overline{\alpha}_1 + \xi_\star(1+\epsilon) + \tfrac{\overline{\alpha}_1}{p_1}\bar{\delta}_1 + \mu_0(|x(0)| + |\delta|_\infty, t) \text{ for all } t \ge T_d^\sharp(|\xi(0)|). \tag{32}$$

Also, by adding and subtracting $\mathrm{sat}_{p_1}(\eta(\ell)x_1(t))$ in its integrand, (25) implies that for all $t \ge T_d^\sharp(|\xi(0)|)$,

$$\dot{x}_1(t) \quad = \quad -\tfrac{k^2\overline{\alpha}_1}{(1-e^{-kT})^2 p_1}\int_{t-T}^{t}\int_{m-T}^{m} e^{k(\ell - t)}\mathrm{sat}_{p_1}(\eta(\ell)x_1(t))\,\mathrm{d}\ell\mathrm{d}m + \mathcal{R}_2(t) + \mathcal{J}_1(t), \text{ where} \tag{33a}$$

$$\mathcal{R}_2(t) = -\frac{k^2\overline{\alpha}_1}{(1-e^{-kT})^2 p_1}\int_{t-T}^{t}\int_{m-T}^{m} e^{k(\ell-t)}\left[\mathrm{sat}_{p_1}(\eta(\ell)x_1(\sigma(\ell))) - \mathrm{sat}_{p_1}(\eta(\ell)x_1(t))\right]\mathrm{d}\ell\mathrm{d}m, \tag{33b}$$

and where we can use the first equality in (10) and the global Lipschitz constant of 1 for sat_{p_1} to obtain

$$|\mathcal{R}_2(t)| \quad \le \quad \tfrac{k^2\overline{\alpha}_1}{(1-e^{-kT})^2 p_1}\overline{\eta}\int_{t-T}^{t}\int_{m-T}^{m} e^{k(\ell-t)}|x_1(t) - x_1(\sigma(\ell))|\,\mathrm{d}\ell\mathrm{d}m \quad \le \quad \frac{\overline{\alpha}_1\overline{\eta}}{p_1}\int_{t-2T-\overline{\sigma}}^{t}|\dot{x}_1(s)|\mathrm{d}s, \tag{34}$$

since the Fundamental Theorem of Calculus gives $|x_1(t) - x_1(\sigma(\ell))| \le \int_{t-2T-\overline{\sigma}}^{t}|\dot{x}_1(s)|\mathrm{d}s$ for all $\ell \in [t-2T-\bar{\sigma}, t]$.

Although we can use the bound L_1 on $\dot{x}_1$ from (1) and (34) to upper bound $|\mathcal{R}_2(t)|$ by $(\bar{\alpha}_1\bar{\eta}/p_1)(2T+\bar{\sigma})L_1$, we will instead use (32) to upper bound $|\dot{x}_1(s)|$ in (34) in order to obtain a bound on $\mathcal{R}_2$ that contains $\bar{\delta}_1$ that is needed to prove our input-to-state stability estimate. To this end, notice that from (32), it follows that for all $t \ge T_d^\sharp(|\xi(0)|) + 2T + \overline{\sigma}$, we have

$$|\mathcal{R}_2(t)| \quad \le \quad \tfrac{\overline{\alpha}_1\overline{\eta}}{p_1}(2T+\overline{\sigma})\left(\overline{\alpha}_1 + \xi_\star(1+\epsilon) + \mu_0^\flat(|x(0)| + |\delta|_\infty, t) + \tfrac{\overline{\alpha}_1}{p_1}\bar{\delta}_1\right), \tag{35}$$

where $\mu_0^\flat(s,t) = \mu_0(s, t - 2T - \overline{\sigma})$. Combining (35) and the second inequality in (20) with (31), we obtain a class $\mathcal{M}$ function $T_d^{\sharp\sharp} : [0, +\infty) \to [0, +\infty)$ such that

$$\sup_{t \ge T_d^{\sharp\sharp}(|\xi(0)|)}(|\mathcal{R}_2(t)| + |\mathcal{J}_1(t)|) < \tfrac{\overline{\alpha}_1}{\overline{\eta}}, \tag{36}$$

where the last inequality is a consequence of (7h) and by choosing $\epsilon \in (0,1)$ to be small enough. Hence, there is a class $\mathcal{M}$ function $T_e : [0, +\infty) \to [0, +\infty)$ such that $\overline{\eta}|x_1(t)| < p_1$ and so also $|\eta(\ell)x_1(t)| < p_1$ for all $t \ge T_e(|\xi(0)|)$ and $\ell \ge 0$. It follows from (33a) that

$$\dot{x}_1(t) \quad = \quad -\tfrac{k^2\overline{\alpha}_1}{(1-e^{-kT})^2 p_1}\int_{t-T}^{t}\int_{m-T}^{m} e^{k(\ell-t)}\eta(\ell)x_1(t)\mathrm{d}\ell\mathrm{d}m + \mathcal{R}_2(t) + \mathcal{J}_1(t) \tag{37}$$

for all $t \geq T_e(|\xi(0)|)$. This follows from applying Lemma 3 to the system (33a), and choosing $q = x_1$ and $\bar{M} = p_1/\bar{\eta}$ in the lemma, because $|\xi| \geq |x_1|$. We can assume that $T_e(s) \geq T_d^\sharp(s) + 2T + \bar{\sigma}$ for all $s \geq 0$.

Let us consider the candidate Lyapunov function $\nu(x_1) = \frac{1}{2}x_1^2$. Since η is lower bounded by 1, we deduce from the second inequality in (20) and (34) that

$$\begin{array}{rcl} \dot{\nu}(t) & \leq & -\frac{\overline{\alpha}_1}{p_1}x_1^2(t) + x_1(t)\left[\mathcal{R}_2(t) + \mathcal{J}_1(t)\right] \\ & \leq & -\frac{\overline{\alpha}_1}{p_1}x_1^2(t) + |x_1(t)|\frac{\overline{\alpha}_1\overline{\eta}}{p_1}\int_{t-2T-\overline{\sigma}}^t |\dot{x}_1(s)|\mathrm{d}s + |x_1(t)|\left(|\xi_2(t)| + \frac{\overline{\alpha}_1}{p_1}\overline{\delta}_1\right) \end{array} \tag{38}$$

holds along all solutions of (37) for all $t \geq T_e(|\xi(0)|)$, by the first equality in (10). From (25) and the first equality in (10), we deduce that

$$\begin{array}{rcl} |\dot{x}_1(t)| & \leq & \frac{k^2\overline{\alpha}_1}{(1-e^{-kT})^2 p_1}\int_{t-T}^t\int_{m-T}^m e^{k(\ell-t)}\overline{\eta}|x_1(\sigma(\ell))|\mathrm{d}\ell\mathrm{d}m + |\mathcal{J}_1(t)| \\ & \leq & \frac{\overline{\alpha}_1\overline{\eta}}{p_1}\sup\limits_{m\in[t-2T-\overline{\sigma},t]}|x_1(m)| + |\xi_2(t)| + \frac{\overline{\alpha}_1}{p_1}\overline{\delta}_1 \end{array} \tag{39}$$

for all $t \geq T_e(|\xi(0)|)$, by our bound $\bar{\eta}$ on η, and where the second inequality is by (20). Hence, (38) gives

$$\begin{array}{rcl} \dot{\nu}(t) & \leq & -\frac{\overline{\alpha}_1}{p_1}x_1^2(t) + |x_1(t)|\frac{\overline{\alpha}_1\overline{\eta}}{p_1}\int_{t-2T-\overline{\sigma}}^t\left(\frac{\overline{\alpha}_1\overline{\eta}}{p_1}\sup\limits_{m\in[s-2T-\overline{\sigma},s]}|x_1(m)| + |\xi_2(s)|\right)\mathrm{d}s \\ & & +|x_1(t)|\left(\frac{\overline{\alpha}_1\overline{\eta}}{p_1}(2T+\overline{\sigma})\frac{\overline{\alpha}_1}{p_1}\overline{\delta}_1 + |\xi_2(t)| + \frac{\overline{\alpha}_1}{p_1}\overline{\delta}_1\right) \\ & \leq & -\frac{\overline{\alpha}_1}{p_1}x_1^2(t) + \frac{\overline{\alpha}_1^2\overline{\eta}^2}{p_1^2}(2T+\overline{\sigma})\sup\limits_{m\in[t-4T-2\overline{\sigma},t]}|x_1(m)|^2 \\ & & +|x_1(t)|\left[\frac{\overline{\alpha}_1\overline{\eta}}{p_1}\int_{t-2T-\overline{\sigma}}^t|\xi_2(s)|\mathrm{d}s + \frac{\overline{\alpha}_1^2\overline{\eta}}{p_1^2}(2T+\overline{\sigma})\overline{\delta}_1 + |\xi_2(t)| + \frac{\overline{\alpha}_1}{p_1}\overline{\delta}_1\right] \end{array} \tag{40}$$

for all $t \geq T_e(|\xi(0)|) + 2T + \bar{\sigma}$. Setting $\xi_\star^\sharp = \xi_\star(1+\epsilon)$, it follows from (31) that, for all $t \geq T_e(|\xi(0)|) + 2T + \bar{\sigma}$,

$$\begin{array}{rl} \dot{\nu}(t) \leq & -\frac{\overline{\alpha}_1}{p_1}x_1^2(t) + \frac{\overline{\alpha}_1^2\overline{\eta}^2}{p_1^2}(2T+\overline{\sigma})\sup\limits_{m\in[t-4T-2\overline{\sigma},t]}|x_1(m)|^2 \\ & +\{|x_1(t)|\}\left\{\frac{\overline{\alpha}_1\overline{\eta}}{p_1}(2T+\overline{\sigma})(\xi_\star^\sharp + \mu_0(|x(0)|+|\delta|_\infty,t)) + \frac{\overline{\alpha}_1^2\overline{\eta}(2T+\overline{\sigma})\overline{\delta}_1}{p_1^2} + \xi_\star^\sharp + \mu_0(|x(0)|+|\delta|_\infty,t) + \frac{\overline{\alpha}_1\overline{\delta}_1}{p_1}\right\} \\ \leq & -\frac{2(1-\omega_0)\overline{\alpha}_1}{p_1}\nu(x_1(t)) + \frac{2\overline{\alpha}_1^2\overline{\eta}^2}{p_1^2}(2T+\overline{\sigma})\sup\limits_{m\in[t-4T-2\overline{\sigma},t]}\nu(x_1(m)) \\ & +\frac{p_1}{4\omega_0\overline{\alpha}_1}\left[\frac{\overline{\alpha}_1\overline{\eta}}{p_1}(2T+\overline{\sigma})(\xi_\star^\sharp + \mu_0(|x(0)|+|\delta|_\infty,t)) + \frac{\overline{\alpha}_1^2\overline{\eta}(2T+\overline{\sigma})\overline{\delta}_1}{p_1^2} + \xi_\star^\sharp + \mu_0(|x(0)|+|\delta|_\infty,t) + \frac{\overline{\alpha}_1}{p_1}\overline{\delta}_1\right]^2 \end{array} \tag{41}$$

where the last inequality applied Young's inequality $ab \leq \frac{\overline{\alpha}_1\omega_0}{p_1}a^2 + \frac{p_1}{4\overline{\alpha}_1\omega_0}b^2$ to the terms in curly braces.

By (7e), we can assume that $\omega_0 \in (0,1)$ is small enough so that

$$\frac{(1-\omega_0)\overline{\alpha}_1}{p_1} > \frac{\overline{\alpha}_1^2\overline{\eta}^2}{p_1^2}(2T+\overline{\sigma}).$$

From Lemma 4 (with $\mathfrak{c}_1 = 2(1-\omega_0)\bar{\alpha}_1/p_1$ and $\mathfrak{c}_2 = 2\bar{\alpha}_1^2\bar{\eta}^2(2T+\bar{\sigma})/p_1^2$) and the second inequality in (41), there is a $\mu_1 \in \mathcal{KL}$ and a class $\mathcal{M}$ function $T_e^\sharp : [0,+\infty) \to [0,+\infty)$ such that for all $t \geq T_e^\sharp(|\xi(0)|)$, we have

$$\nu(x_1(t)) \leq \left\{\frac{p_1^3(1+\omega_0)}{8\overline{\alpha}_1^2\omega_0(p_1(1-\omega_0)-\overline{\alpha}_1\overline{\eta}^2(2T+\overline{\sigma}))}\left[\frac{\overline{\alpha}_1\overline{\eta}(2T+\overline{\sigma})\xi_\star^\sharp}{p_1} + \frac{\overline{\alpha}_1^2\overline{\eta}(2T+\overline{\sigma})\overline{\delta}_1}{p_1^2} + \xi_\star^\sharp + \frac{\overline{\alpha}_1\overline{\delta}_1}{p_1}\right]^2\right\} + \mu_1(|x(0)|+|\delta|_\infty,t), \tag{42}$$

by the argument that led to (31) and the linear growth of the x_1 dynamics.

3.4.3 Third Step: Deriving Bounds on the Original State Variables

Using (42), it follows that with the choice $\mu_2 = \sqrt{2\mu_1}$, we get

$$|x_1(t)| \leq \gamma_a + \mu_2(|x(0)|+|\delta|_\infty,t), \tag{43}$$

where $\gamma_a = \sqrt{2B_a}$ and B_a is the quantity in curly braces in (42), by the subadditivity of the square root and the formula $\nu(x_1) = \frac{1}{2}x_1^2$. We next find analogous bounds for x_2 and x_3.

From (31) and the definition $\xi_2 = x_2 - \alpha$ from (14), and also using (9b), we deduce that

$$\left| x_2(t) + \tfrac{k^2}{(1-e^{-kT})^2} \int_{t-T}^{t} \int_{m-T}^{m} e^{k(\ell-t)} \frac{\overline{\alpha}_1}{p_1} \mathrm{sat}_{p_1}(\eta(\ell)x_1(\sigma(\ell)) + \delta_1(\ell)) \mathrm{d}\ell \mathrm{d}m \right| \quad \leq \quad \xi_\star^\sharp + \mu_0(|x(0)| + |\delta|_\infty, t) \tag{44}$$

and so also

$$|x_2(t)| \quad \leq \quad \xi_\star^\sharp + \mu_0(|x(0)| + |\delta|_\infty, t) + \tfrac{k^2}{(1-e^{-kT})^2} \tfrac{\overline{\alpha}_1}{p_1} \int_{t-T}^{t} \int_{m-T}^{m} e^{k(\ell-t)} [\overline{\eta}|x_1(\sigma(\ell)| + \overline{\delta_1}] \mathrm{d}\ell \mathrm{d}m \tag{45}$$

for all $t \geq T_d^\sharp(|\xi(0)|)$, since $|\mathrm{sat}_{p_1}(r)| \leq r$ for all $r \geq 0$. From (45) and (43) and the first equality in (10), we deduce that for all $t \geq T_e^\sharp(|\xi(0)|) + 2T + \bar{\sigma}$, we have

$$|x_2(t)| \quad \leq \quad \xi_\star^\sharp + \mu_0^\sharp(|x(0)| + |\delta|_\infty, t) + \tfrac{\overline{\alpha}_1}{p_1} \left[\overline{\eta}\gamma_a + \overline{\delta_1} \right] \tag{46}$$

where $\mu_0^\sharp = \mu_0 + (\bar{\alpha}_1/p_1)\overline{\eta}\mu_2 \in \mathcal{KL}$. Moreover, since $x_3(t) = \dot{\alpha}(t) + \beta(t) + \xi_3(t)$, we have

$$\begin{aligned} |x_3(t)| \quad \leq \quad & \Big[\tfrac{k^3}{(1-e^{-kT})^2} \int_{t-T}^{t} \int_{m-T}^{m} e^{k(\ell-t)} |\phi(\ell)| \mathrm{d}\ell \mathrm{d}m + \tfrac{k^2}{(1-e^{-kT})^2} \int_{t-T}^{t} e^{k(\ell-t)} |\phi(\ell)| \mathrm{d}\ell \\ & + \tfrac{k^2}{(1-e^{-kT})^2} \int_{t-2T}^{t-T} e^{k(\ell-t)} |\phi(\ell)| \mathrm{d}\ell \Big] + \tfrac{k}{1-e^{-kT}} \int_{t-T}^{t} e^{k(\ell-t)} |\omega(\ell)| \mathrm{d}\ell + |\xi_3(t)| \end{aligned} \tag{47}$$

for all $t \geq 2T$, by (9b), where we note (for later use in the proof) that the quantity in squared brackets in (47) is an upper bound for $|\dot{\alpha}_1(t)|$. Next observe that our choices of y_1 and y_2 from (2) give

$$\begin{aligned} & |\phi(t)| \leq \tfrac{\overline{\alpha}_1}{p_1} \big| \eta(t)x_1(\sigma(t)) + \delta_1(t) \big| \\ & \text{and} \;\; |\omega(t)| \leq \tfrac{\overline{\beta}_1}{p_2} \left| x_2(\sigma(t)) + \delta_2(t) + \tfrac{k^2}{(1-e^{-kT})^2} \int_{\sigma(t)-T}^{\sigma(t)} \int_{m-T}^{m} e^{k(\ell-\sigma(t))} \phi(\ell) \mathrm{d}\ell \mathrm{d}m \right| \end{aligned} \tag{48}$$

for all $t \geq 2T + \bar{\sigma}$. It follows from (43) and (46) that we can find a function $\mu_3 \in \mathcal{KL}$ such that

$$|\phi(t)| \leq \phi_\star + \mu_3(|x(0)| + |\delta|_\infty, t), \;\; \text{where} \;\; \phi_\star = \tfrac{\overline{\alpha}_1}{p_1} \left(\overline{\eta}\gamma_a + \overline{\delta}_1 \right) \tag{49}$$

$$\begin{aligned} & \text{and} \quad |\omega(t)| \leq \tfrac{\overline{\beta}_1}{p_2} \left[\xi_\star^\sharp + \tfrac{\overline{\alpha}_1}{p_1} \left[\overline{\eta}\gamma_a + \overline{\delta_1} \right] \right] \\ & + \tfrac{\overline{\beta}_1}{p_2} \left(\mathfrak{s} \int_{\sigma(t)-T}^{\sigma(t)} \int_{m-T}^{m} e^{k(\ell-\sigma(t))} \mathrm{d}\ell \mathrm{d}m \phi_\star + \overline{\delta}_2 + \mu_3(|x(0)| + |\delta|_\infty, t) \right) \;\; \leq \;\; \omega_\star + \tfrac{\overline{\beta}_1}{p_2} \mu_3(|x(0)| + |\delta|_\infty, t), \end{aligned} \tag{50}$$

by enlarging $T_e^\sharp$ as needed without relabeling and using the first equality in (10), where $\mathfrak{s} = k^2/(1-e^{-kT})^2$ and where

$$\omega_\star = \tfrac{\overline{\beta}_1}{p_2} \left[\xi_\star^\sharp + \tfrac{\overline{\alpha}_1}{p_1} (\overline{\eta}\gamma_a + \overline{\delta_1}) + \phi_\star + \overline{\delta}_2 \right]. \tag{51}$$

Also, by using (10) and the upper bound on $|\dot{\alpha}_1(t)|$ that we obtained in squared brackets in (47), we have $|\dot{\alpha}(t)| \leq 2k|\phi|_{[t-2T,t]}/(1-e^{-kT})$ for all $t \geq 2T$. Therefore, we deduce from the formula $x_3(t) = \dot{\alpha}(t) + \beta(t) + \xi_3(t)$ from (14) and from our bounds on ϕ and ω from (49) and (50) (and the asymptotic stability property of the ξ_3 subsystem) that we can find a class $\mathcal{M}$ function $T_f : [0, +\infty) \to [0, +\infty)$ and a function $\mu_4 \in \mathcal{KL}$ such that

$$|x_3(t)| \leq \tfrac{2k}{1-e^{-kT}} \phi_\star + \omega_\star + \mu_4(|x(0)| + |\delta|_\infty, t) \tag{52}$$

for all $t \geq T_f(|\xi(0)|)$, using the formula for β from (9b) and the second equality in (10). The theorem now follows from combining the upper bounds (43), (46), and (52); see Appendix 4 below.

4 Application to Visual Landing of Aircraft

To illustrate our results, we consider the lateral dynamics of an Airbus airliner in a glide phase which must align with a runway using a body fixed monocular camera [2], using imprecise, sampled, or delayed output measurements. This problem is a challenge of strong relevance in cases where the runway is unequipped or in the case of GPS loss, where the output delays and sampling arise from image processing. More precisely, the position, size and heading of the runway are unknown, so the relative position (Δ_X, Δ_Y) and heading Δ_ψ of the aircraft with respect to it are unmeasured. See Figure 1, and [10] for valuable research on vision based aircraft control, which does not provide the input-to-state stability to uncertainty under sampling that we provide here.

As noted in [13], this research was motivated by this simplified lateral guidance model provided by Airbus:

$$\left\{\begin{array}{rcl} \dot{\Delta}_Y & = & V\mathrm{sat}_{L_\psi}(\Delta_\psi) \\ \dot{\Delta}_\psi & = & \frac{g}{V}\mathrm{sat}_{L_\varphi}(\varphi) \\ \dot{\varphi} & = & \mathrm{sat}_{L_u}(u_{\mathrm{lat}}), \end{array}\right. \tag{53}$$

where u_{lat} is the input, $V = 72\mathrm{m.s}^{-1}$ is constant all along the final approach, $g = 9.81\mathrm{m.s}^{-2}$, and φ (resp., u_{lat}) is the aircraft roll angle (resp., the guidance/outer loop control action). Then (53) can be transformed into the system (1) by applying the changes of coordinates $x_1 = \Delta_Y$, $x_2 = V\Delta_\psi$, and $x_3 = g\varphi$ and setting $L_1 = VL_\psi$, $L_2 = gL_\varphi$, $L_3 = gL_u$, and $u = gu_{\mathrm{lat}}$. Here, we assume that we can extract the quantities

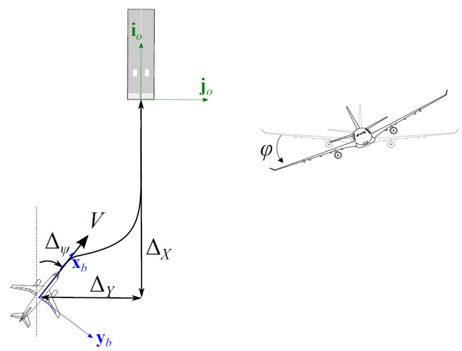

Figure 1: Notation used in the alignment part of the glide phase

$y_1 = \eta(t)\Delta_Y(\sigma(t)) + \delta_1(t)$, $y_2 = \Delta_\psi(\sigma(t)) + \delta_2(t)$, and $y_3 = \varphi(t)$ from the images and inertial measurement unit, where $\eta(t) \in [1, 2]$ and $\delta_1(t)$ and $\delta_2(t)$ represent the measurement noise, which are mainly due to image acquisition and processing, and $\sigma(t)$ is due to the fact that the images are sampled and processed. Therefore, we choose $\bar{\eta} = 2$.

The saturation limits are $L_1 = 25\mathrm{m.s}^{-1}$, $L_2 = 7\mathrm{m.s}^{-2}$, and $L_3 = 6\mathrm{m.s}^{-3}$. With the preceding choices, the values

$$k = 0.1,\ T = 2,\ L_4 = 1,\ \bar{\alpha}_1 = 1.23,\ p_1 = 24,\ \bar{\beta}_1 = 0.2,\ \text{and}\ p_2 = 2.5 \tag{54}$$

satisfy (7) from Theorem 1, when $\bar{\sigma}$, $\bar{\delta}_1$, and $\bar{\delta}_2$ are chosen as follows, where 878ms in the first row of the table and 5.5 in the third row of the table corresponds to choosing $\bar{\sigma} = 0.878$ and $\bar{\delta}_1 = 0.055$ respectively in our conditions (7) on our constants (and similarly for the other rows):

$\bar{\sigma}$ (ms)	$\bar{\delta}_1$ (cm)	$\bar{\delta}_2$ (cm/s)
878	0	0
0	152.5	0
0	0	5.5
100	67.02	2.4

Table 1: Maximum allowable values according to our main result for the parameter values (54).

In Table 1, the first line gives the maximum value of $\bar{\sigma}$, in the special case $\bar{\delta}_1 = \bar{\delta}_2 = 0$ that we discussed in Section 3.2 (corresponding to choosing $\bar{\sigma} = 0.878$ in (7) because of the scaling of the physical quantities). Then the next two lines show maximum disturbance values in the special case that we discussed in Section 3.2 where $\bar{\sigma} = 0$, which corresponds to the undelayed case where $\sigma(t) = t$ where there is no sampling in the outputs. In practice, assuming that image processing and acquisition is done within 878ms seems reasonable (since, for instance, some image processing algorithms run at around 67ms in [6]); indeed, once correctly initialized, the computer vision algorithm must simply track the runway (which is a trapezoid) in the image. The second (resp. third) line gives the maximum value of $\bar{\delta}_1$ (resp., $\bar{\delta}_2$) assuming $\bar{\sigma} = 0$ and $\bar{\delta}_2 = 0$ (resp., $\bar{\delta}_1 = 0$). Finally, we can

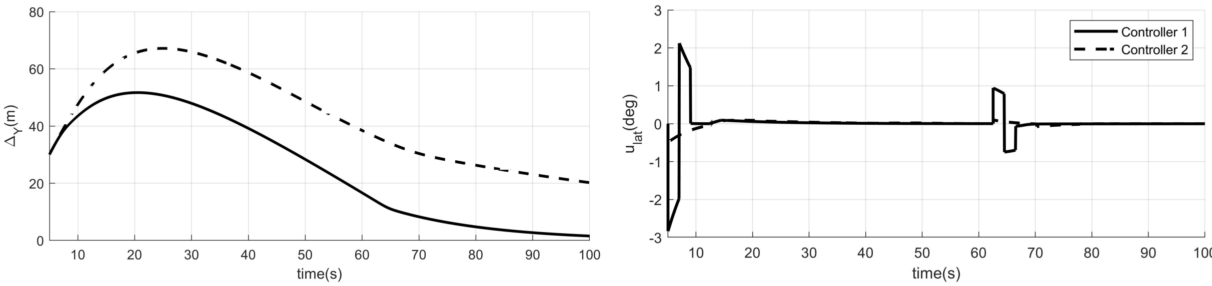

Figure 2: State Δ_Y and control u_{lat} using values from last line of Table 1.

determine maximal allowable values when all of these values are nonzero, as follows. Assume that the image processing and acquisition are performed at 100 ms. Then, a good compromise is in the last line, in which $\bar{\delta}_1$ and $\bar{\delta}_2$ are selected such that (7) hold with $\bar{\sigma} = 100$ms, $\bar{\delta}_1 = 1.525x\,$m, and $\bar{\delta}_2 = .055x\,$m/s, where the value $x = 0.4395$ was selected as the largest x such that (7) hold when $\bar{\delta}_1 = 1.525x\,$m, and $\bar{\delta}_2 = .055x\,$m/s. This is a compromise, because $\bar{\sigma}$ is reduced from 878ms, to allow positive $\bar{\delta}_i$ values that agree with a suitable percent of their maximum values. Moreover, for the values $\bar{\sigma} = 0$, $\bar{\delta}_1 = \bar{\delta}_2 = 0.27$, $k = 0.1$, $T = 2$, $L_4 = 1$, $\bar{\alpha}_1 = 1.6$, $p_1 = 69$, $\bar{\beta}_1 = 0.2$, and $p_2 = 2.5$, the assumptions of Theorem 1 are again satisfied, but the assumptions of [14] would not be satisfied, because the preceding values give $L_1(1-e^{-1})^2/(40(1+2e^{-1}+e^{-2})) - \bar{\delta}_2 < 0$, so Theorem 1 is less restrictive than [14].

For comparison purposes, we name our guidance solution from Theorem 1 'Controller 1'. For the plots above, we used the values of $\bar{\sigma}$ and $\bar{\delta}_1$ and $\bar{\delta}_2$ calculated in the last row of the table, with the choice $\sigma(t) = i\bar{\sigma}/2 - \bar{\sigma}/2$ for all $t \in [i\bar{\sigma}/2, (i+1)\bar{\sigma}/2)$ and $i \geq 0$ (to incorporate the effects of sampling and delay). One can also apply a second controller called Controller 2 to (53) based on [14], by choosing the parameters in [14] to be $q = 0.07$ (which is the parameter of [14, Remark 1]), $r = 0.13$, $\lambda = \lambda_a = 0.007$, $\bar{\sigma}_\star = 0.07$, and $\epsilon = 0.01$. These choices of parameters give the value $\bar{v} = 0.63$ for the corresponding parameter in [14]. We were unable to find larger values forλ or r while respecting the assumptions of the main result of [14]; these values have a direct impact on controller performance. Figure 2 shows how Controller 1 succeeds to lower a lateral deviation of 30m for an initial heading deviations $\Delta_\psi = 3$deg within 42s. Moreover, Controller 1 (in terms of radians) has the bound

$$L_4 + \bar{\alpha}_3 + \bar{\beta}_2 = L_4 + \tfrac{4k^2\bar{\alpha}_1}{(1-e^{-kT})^2} + \tfrac{2k\bar{\beta}_1}{1-e^{-kT}} = 1 + \tfrac{4(0.1)^2(1.23)}{(1-e^{-0.1(2)})^2} + \tfrac{2(0.1)(0.2)}{1-e^{-0.1(2)}} = 2.718 \tag{55}$$

that is ensured by our theorem. We start the plots at time $t = 4$ because Controller 1 from (4a) is 0 on the interval $[0, 2T] = [0, 4]$. In the right panel of Figure 2, we see that $|u_{\text{lat}}|$ (expressed in degrees per second) stays below its required bound $(180/(\pi g)) \times 2.718 = 15.87$. It also shows that the Controller 2 is unable to reduce the lateral deviation in such a short time. This indicates that the compromise between robustness and performance is worse for Controller 2. In other words, the conditions for the main result of [14] are more conservative.

5 Conclusions

We used a recent backstepping approach to derive a useful new class of bounded controls for a chain of saturated integrators that arises in the visual landing of aircraft under image processing. This overcame the challenge of having imprecise, delayed, or sampled output measurements by proving an input-to-state stability estimate. Our numerical simulations illustrated potential advantages of this work over previous methods. In future work, we hope to merge our results with existing methods for approximating delays to cover cases where σ may not be known, and to transition this work to practice to improve the performance of aircraft during the glide phase.

Appendix 1: Proof of Lemma 1

Simple calculations (based on substituting (6a)-(6b) into (5)) show that the functions v_i from (4b)-(4d) satisfy $v_1(z_t) = \dot{\alpha}(t)$, $v_2(y_t, z_t) = \ddot{\alpha}(t)$, and $v_3(y_t, z_t) = \dot{\beta}(t)$ for all $t \geq 0$. This allows us to rewrite (4a) as (9a). Also, e can apply variation of parameters on the interval $[t-T, t]$ (to the system $\dot{q} = k(-q+b)$ with the choice $(q(t), b(t)) = (z_3(t), \omega(t))$, then with the choice $(q(t), b(t)) = (z_2(t), z_1(t) - e^{-kT}z_1(t-T))$, and finally with the choice $(q(t), b(t)) = (z_1(t), \phi(t)))$ to prove that (9b) holds for all $t \geq 2T$. It remains to prove the bounds (9c).

To this end, we use (9b) to prove the bounds on the derivatives of α and β in (9c); the bounds on α and β in (9c) follow from the definition of the saturation and (9b) and (10). We continue to use the common notation $\mathfrak{s} = k^2/(1-e^{-kT})^2$ and the equalities and inequalities to follow should be understood to hold for all $t \geq 2T$. We have

$$\dot{\alpha}(t) \;=\; -k\alpha(t) + \mathfrak{s}\left[\int_{t-T}^{t} e^{k(\ell-t)}\phi(\ell)\mathrm{d}\ell - \int_{t-2T}^{t-T} e^{k(\ell-t)}\phi(\ell)\mathrm{d}\ell\right], \tag{A.1}$$

and therefore also

$$|\dot{\alpha}|_\infty \;\leq\; k\overline{\alpha}_1 + \mathfrak{s}\overline{\alpha}_1\left[\tfrac{1}{k}\left(1-e^{-kT}\right) + \tfrac{1}{k}\left(e^{-kT} - e^{-2kT}\right)\right] \;=\; k\overline{\alpha}_1 + \mathfrak{s}\overline{\alpha}_1\tfrac{1}{k}\left(1-e^{-kT}\right)\left(1+e^{-kT}\right) \;=\; \overline{\alpha}_2, \tag{A.2}$$

by our choice of $\overline{\alpha}_2$ in (8). Also, since

$$\begin{aligned}\ddot{\alpha}(t) \;&=\; -k\dot{\alpha}(t) + \mathfrak{s}(-k)\left[\int_{t-T}^{t} e^{k(\ell-t)}\phi(\ell)\mathrm{d}\ell - \int_{t-2T}^{t-T} e^{k(\ell-t)}\phi(\ell)\mathrm{d}\ell\right]\\ &\quad + \mathfrak{s}[\phi(t) - 2e^{-kT}\phi(t-T) + e^{-2kT}\phi(t-2T)],\end{aligned} \tag{A.3}$$

our formulas for $\overline{\alpha}_2$ and $\overline{\alpha}_3$ from (8) and the bound $\overline{\alpha}_1$ on ϕ give

$$|\ddot{\alpha}|_\infty \;\leq\; \tfrac{2k^2\bar{\alpha}_1}{1-e^{-kT}} + \left[\mathfrak{s}\left(1-e^{-kT}\right) + \mathfrak{s}\left(e^{-kT} - e^{-2kT}\right)\right]\bar{\alpha}_1 + \mathfrak{s}\left(1+2e^{-kT}+e^{-2kT}\right)\bar{\alpha}_1 \;\leq\; 4\overline{\alpha}_1\mathfrak{s} \;=\; \overline{\alpha}_3. \tag{A.4}$$

Finally, since

$$\dot{\beta}(t) \;=\; -k\beta(t) + \tfrac{k}{1-e^{-kT}}\left(\omega(t) - \omega(t-T)e^{-kT}\right), \tag{A.5}$$

we have

$$|\dot{\beta}|_\infty \;\leq\; k\overline{\beta}_1 + \tfrac{k}{1-e^{-kT}}\left(1+e^{-kT}\right)\overline{\beta}_1 \;=\; \overline{\beta}_2, \tag{A.6}$$

by our choice of $\overline{\beta}_2$ in (8) and our bound $\overline{\beta}_1$ on ω, which completes our proof of the bounds (9c).

Appendix 2: Proof of Lemma 3

Consider any solution $q : [T_0,\infty) \to \mathbb{R}$ of (11). If there is a time $t_* \geq \bar{T}$ where $|q(t_*)| \leq \bar{M}$, and a time $t > t_*$ and an $\epsilon > 0$ such that $|q(t)| \geq \bar{M} + \epsilon$, then let $T_{\min}$ be the smallest time $t > t_*$ such that $|q(t)| \geq \bar{M} + \epsilon$. Then, if $q(T_{\min}) > 0$, then $\dot{q}(T_{\min}) = -\mathcal{G}(T_{\min}, q(T_{\min})) + H(T_{\min}) \leq -g_0 + \bar{H} < 0$, which implies that there is a smaller time $t \in (t_*, T_{\min})$ at which $q(t) \geq \bar{M} + \epsilon$, which contradicts the minimality of $T_{\min}$. On the other hand, if $q(T_{\min}) < 0$, then $\dot{q}(T_{\min}) = -\mathcal{G}(T_{\min}, q(T_{\min})) + H(T_{\min}) \geq g_0 - \bar{H} > 0$, so there is a smaller time $t \in (t_*, T_{\min})$ at which $q(t) \leq -\bar{M} - \epsilon$, which again contradicts the minimality of $T_{\min}$.

The preceding argument implies that for each $t \geq \bar{T}$ such that $|q(t)| \leq \bar{M}$, we have $|q(s)| \leq \bar{M}$ for all $s \geq t$. It follows that if $t \geq \bar{T}$ is such that $|q(t)| > \bar{M}$, then $|q(r)| > \bar{M}$ for all $r \in [\bar{T}, t]$. Therefore, if $t \geq \bar{T}$ is a time when $q(t) > \bar{M}$, then $\dot{q}(\ell) \leq -g_0 + \bar{H}$ for all $\ell \in [\bar{T}, t]$, so applying the Fundamental Theorem of Calculus to q on the interval $[\bar{T}, t]$ gives $0 < \bar{M} \leq q(t) \leq q(\bar{T}) + (\bar{H} - g_0)(t - \bar{T}) \leq |q(T_0)| + (|\mathcal{G}|_\infty + |\mathcal{H}|_\infty)(\bar{T} - T_0) + (\bar{H} - g_0)(t - \bar{T})$, which implies that

$$t \leq \tfrac{|q(T_0)| + (|\mathcal{G}|_\infty + |\mathcal{H}|_\infty)(\bar{T} - T_0)}{g_0 - \bar{H}} + \bar{T}. \tag{A.7}$$

Similarly, if $t \geq \bar{T}$ is a time when $q(t) < -\bar{M}$, then $\dot{q}(\ell) \geq g_0 - \bar{H}$ for all $\ell \in [\bar{T}, t]$, so we have $0 > -\bar{M} \geq q(t) \geq q(\bar{T}) - (\bar{H} - g_0)(t - \bar{T}) \geq -|q(T_0)| - (|\mathcal{G}|_\infty + |\mathcal{H}|_\infty)(\bar{T} - T_0) - (\bar{H} - g_0)(t - \bar{T})$, which again gives (A.7). Hence, by separately considering the cases $|q(\bar{T})| \leq \bar{M}$ and $|q(\bar{T})| > \bar{M}$, it follows that we can choose

$$T_*(s) = \tfrac{s + (|\mathcal{G}|_\infty + |\mathcal{H}|_\infty)(\bar{T} - T_0)}{g_0 - \bar{H}} + \bar{T} \tag{A.8}$$

to satisfy our requirements, because $|q(t)| \leq \bar{M}$ for all $t \geq T_*(|q(T_0)|)$.

Appendix 3: Proof of Lemma 4

First, observe that $\frac{\Delta_1}{\mathfrak{c}_1 - \mathfrak{c}_2}$ is well-defined because $\mathfrak{c}_1 > \mathfrak{c}_2$. Let $\tilde{v}(t) = v(t) - \frac{\Delta_1}{\mathfrak{c}_1 - \mathfrak{c}_2}$. Then

$$\dot{\tilde{v}}(t) \leq -\mathfrak{c}_1\tilde{v}(t) + \mathfrak{c}_2 \sup_{m \in [t-h,t]} \tilde{v}(m) \tag{A.9}$$

for all $t \geq 0$. Since $\mathfrak{c}_1 > \mathfrak{c}_2$, the required constant c_s exists. Also, the function $p(t) = e^{-c_s t}$ satisfies

$$\dot{p}(t) \quad = \quad -\mathfrak{c}_1 p(t) + \mathfrak{c}_2 \sup_{m \in [t-h,t]} p(m) \tag{A.10}$$

for all $t \geq 0$, by our choice of c_s. It now suffices to prove that $l_v p(t) > \tilde{v}(t)$ for all $t \geq -h$. To this end, suppose that there is a constant $t_c > 0$ such that $l_v p(t) > \tilde{v}(t)$ for all $t \in [-h, t_c)$ and $l_v p(t_c) = \tilde{v}(t_c)$, for the sake of obtaining a contradiction. Let $w(t) = \tilde{v}(t) - l_v p(t)$. Then, using $l_v p(t_c) = \tilde{v}(t_c)$ and (A.9)-(A.10), we get

$$\dot{w}(t_c) \quad \leq \quad \mathfrak{c}_2 \left[\sup_{m \in [t_c-h,t_c]} \tilde{v}(m) - \sup_{m \in [t_c-h,t_c]} l_v p(m) \right] < 0, \tag{A.11}$$

where the last inequality in (A.11) follows because if we choose a $t_* \in [t_c - h, t_c]$ such that $\sup_{m \in [t_c-h,t_c]} \tilde{v}(m) = \tilde{v}(t_*)$, then the quantity in squared brackets in (A.11) is $\tilde{v}(t_*) - l_v p(t_c - h)$, which is negative if $t_* = t_c - h$ (by our choice of t_c) and is also negative if $t_* \in (t_c - h, t_c]$ because in that case it is bounded above by $l_v(p(t_*) - p(t_c - h)) < 0$. From $\dot{w}(t_c) < 0$ and $w(t) < 0$ when $t \in [-h, t_c)$, we deduce that $w(t_c) < 0$, which is a contradiction. Hence, $w(t) < 0$ for all $t \geq -h$. Thus $\tilde{v}(t) < l_v p(t)$ for all $t \geq -h$, which gives the conclusion.

Appendix 4: Completion of Proof of Theorem 1

By combining the upper bounds (43), (46), and (52) for the $|x_i(t)|$'s and recalling the formulas for the components of ξ, we can construct functions $\beta_0 \in \mathcal{KL}$ and $\gamma_0 \in \mathcal{K}_\infty$ and a class $\mathcal{M}$ function $T_g : [0, +\infty) \to [0, +\infty)$ such that

$$|x(t)| \leq \beta_0(|x(0)|, t) + \gamma_0(|\delta|_\infty) \tag{A.12}$$

for all $t \geq T_g(|x(0)|)$, where the construction of T_g used the fact that our formulas (14) imply that there is a class $\mathcal{M}$ function $\Theta : [0, +\infty) \to [0, +\infty)$ such that $|\xi(0)| \leq \Theta(|x(0)|)$ holds for all initial states $\xi(0)$ and $x(0)$, so we can choose $T_g(s) = T_f(\Theta(s))$. Also, the linear growth of the x dynamics provides a constant $\bar{L} > 0$ and a function $\alpha_* \in \mathcal{K}_\infty$ such that

$$|x(t)| \leq \bar{L} e^{\bar{L} T_g(|x(0)|) - t} \alpha_*(|x(0)|) + \bar{L} \alpha_*(|\delta|_\infty) \text{ for all } t \in [0, T_g(|x(0)|)]. \tag{A.13}$$

In fact, (5)-(6b) provide a constant $\bar{c} > 0$ such that $|\dot{z}(t)| \leq \bar{c}(|z|_{[t-T-\bar{\sigma},t]} + |x|_{[t-\bar{\sigma},t]} + |\delta|_\infty)$ and therefore also

$$|z|_{[t-T-\bar{\sigma},t]} \leq \bar{c} \int_0^t \left(|z|_{[\ell-T-\bar{\sigma},\ell]} + |x|_{[\ell-\bar{\sigma},\ell]} \right) \mathrm{d}\ell + \bar{c} t |\delta|_\infty, \tag{A.14}$$

for all $t \geq 0$ (since we assumed that the initial functions for the z dynamics are 0). Therefore, we can apply Gronwall's inequality to the function $\mathcal{F}(t) = |z|_{[t-T-\bar{\sigma},t]}$ to find a constant $\bar{c}^\sharp > 0$ such that

$$|z(t)| \leq \bar{c}^\sharp e^{\bar{c} T_g(|x(0)|)} \left(|x|_{[0,t]} + |\delta|_\infty \right) \tag{A.15}$$

for all $t \in [0, T_g(|x(0)|)]$, which we can use to find positive constants $\tilde{c} > 0$ and $\bar{c}_*$ and a constant $\alpha_a > 0$ so that the right side of (1) is bounded by $\tilde{c}(e^{\bar{c}_* T_g(|x(0)|)} \alpha_a |x|_{[0,t]} + |\delta|_\infty)$ for all $t \in [0, T_g(|x(0)|)]$, and then we can apply Gronwall's inequality to the x system (as we did for the z system) to get the required constant $\bar{L} > 0$. The final input-to-state stability estimate now follows by adding the bounds for (A.12)-(A.13) for $|x(t)|$ to find a bound that holds for all $t \geq 0$.

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